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Painlevé Analysis, Conservation Laws, and Symmetry of Perturbed Nonlinear Equations

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We consider the Lie-Backlund symmetries and conservation laws of a perturbed KdV equation and NLS equation. The arbitrary coefficients of the perturbing terms can be related to the condition of existence of nontrivial LB symmetry generator. When the perturbed KdV equation is subjected to Painlevé analysis *a la* Weiss, it is found that the resonance position changes compared to the unperturbed one. We prove the compatibility of the overdetermined set of equations obtained at the different stages of recursion relations, at least for one branch. All other branches are also indicated and difficulties associated them are discussed considering the perturbation parameter ε to be small. We determine the Lax pair for the aforesaid branch through the use of Schwarzian derivative. For the perturbed NLS equation we determine the conservation laws following the approach of Chen and Liu. From the recurrence of these conservation laws a Lax pair is constructed. But the Painlevé analysis does not produce a positive answer for the perturbed NLS equation. So here we have two contrasting examples of perturbed nonlinear equations: one passes the Painlevé test and its Lax pair can be found from the analysis itself, but the other equation does not meet the criterion of the Painlevé test, though its Lax pair is found in another way.

1. INTRODUCTION

The existence of an infinite number of Lie-Backlund symmetries for partial differential equations that possess a Lax pair is now a proven fact. On the other hand, some perturbed nonlinear equations are also integrable in the sense that they do have a Lax pair up to first order in the perturbation parameter (Kodama, 1985). Here we consider the perturbed KdV and NLS equation.

First we consider the PKdV equation. We see that the different constants occurring in the perturbing term is related to the existence of nontrivial LB symmetries. Encouraged by such an analysis, we then make a Painlevé

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analysis of the equations under consideration, for the further clarification of the complete integrability, though it is known that the Painlevé test is not a necessary and sufficient condition for an equation to be completely integrable (Clarkson, 1987). But in this case of the PKdV under consideration we deduce a Lax pair, which indicates the completely integrable character of the perturbed KdV system. In our all the calculations the parameter ε is small.

Next we examine the perturbed nonlinear Schrödinger equation (PNLS). We first determined the perturbed set of conservation laws. It is then observed that one can set up a recursion operator for these conservation laws and hence can set up a Lax pair *a la* Chen and Liu. But when we perform a Painlevé analysis for PNLS the results are not at all encouraging. In contrast to the case of PKdV, the equations at the resonance positions yield a trivial result and the arbitrary wavefront gets fixed.

2. LIE BACKLUND SYMMETRY OF PERTURBED KdV

The perturbed KdV equation is written as

$$u_+ = u_3 + 6uu_1 \varepsilon(a_1 u_5 + a_2 u u_3 + a_3 u_1 u_2 + a_4 u^2 u_1) \tag{1}$$

A Lie–Backlund transformation is of the form (Fokas and Anderson, 1982)

$$\begin{aligned} u' &= u + \varepsilon\eta(u, u_1, u_2, u_3, \ldots) \\ x' &= x; \qquad t' = t \end{aligned} \tag{2}$$

with ε a small parameter. To first order in ε, the equation to be satisfied by η is (Fuchssteiner and Fokas, 1981)

$$\begin{aligned} \eta_t = {} & \varepsilon a_1 \eta_5 + (\varepsilon a_2 u + 1)\eta_3 + \varepsilon a_3 u_1 \eta_2 + (\varepsilon a_3 u_2 + 6u \\ & + \varepsilon a_4 u^2)\eta_1 + (6u_1 + \varepsilon a_2 u_3 + 2\varepsilon a_4 u u_1)\eta \end{aligned} \tag{3}$$

All the derivatives (space and time) on η are to be interpreted as

$$\eta_+ = \eta_{u_i} u_{it}, \qquad \eta_x = \partial_{x+} u_{i+1}\, \partial\eta/\partial u_i \tag{4}$$

Using (1) and all of its derived consequences for u_{it}, we obtain, by equating coefficients of u_9 in (3),

$$\eta_{a_i u_5} u_{i+1} = 0$$

implying

$$\eta = au + B(u_1, \ldots, u_4) \tag{5}$$

From the coefficients of u_7, u_8, we get

$$\eta = au_5 + 6u_4 + u_3\left(\frac{a^2}{a_1} au + c\right) + E(u_2, \ldots, u) \tag{6}$$

The coefficient of u_6 imposes the condition

$$5a_1(\eta_{u_iu_2}u_{i+1}+2\eta_{u_iu_3}u_{i+2})=4a_2u_1\eta_{u_4}+\eta_{u_5}(10a_2+5a_3)u_2$$

from which we get

$$\eta=au_5+bu_4+u_3\left(\frac{a_2}{a_1}au+c\right)+u_2\left(\frac{4}{5}b\frac{a_2}{a_1}u+\frac{aa_3}{a_1}u_1+d\right)+F(uu_1) \quad (7)$$

Substituting this form of η in the rest of the equations obtained from (3), we get (from the coefficient of u_5)

$$\eta=au_5+bu_4+u_3\left(\frac{a_2a}{a_1}u+c\right)+u_2\left(\frac{4ba_2}{5a_1}u+\frac{aa_3}{a_1}u_1+d\right)$$
$$+\frac{2b}{5a_1}u_1^2(a_3-2a_2)+\frac{u_1}{5a_1}(3a_2cu+5aa_4u^2)$$
$$+\frac{3a}{5\varepsilon a_1}uu_1\left(10-\frac{a_2}{a_1}\right)+eu_1+G \quad (8)$$

The coefficient of u_4 leads to

$$b(4a_2+a_3)=0 \quad (9a)$$

$$\left(c\varepsilon-\frac{a}{a_1}\right)(a_3-2a_2)=0 \quad (9b)$$

$$G=\frac{2b}{15}u_3\left(\frac{2a_4}{a_1}-\frac{a_2^2}{5a_1^2}\right)+u^2\left(\frac{a_2d}{5a_1}+\frac{12b}{5\varepsilon a_1}-\frac{6ba_2}{25\varepsilon a_1^2}\right)+fu+g \quad (9c)$$

Now an important observation about (8) is that it contains a term of the order $1/\varepsilon$, but if we think of ε as smali and want the symmetries of PKdV to go over smoothly to those of KdV as $\varepsilon\to 0$, then this term must vanish and we get $a_2=10a_1$. Equation (9a) also impliex $b=0$. Also, the coefficients of u_4 have the consequences

$$\varepsilon d(a_3-2a_2)+\frac{6b}{5a_1}(3a_2-a_3)-4\varepsilon b\left(2a_4-\frac{a_2^2}{5a_1}\right)=0$$
$$b\left(2a_4-\frac{a_2^2}{5a_1}\right)=0; \qquad a_2d+\frac{12b}{\varepsilon}-\frac{6ba_2}{5\varepsilon a_1}=0 \quad (10)$$

From (9b) we get either $a_3=2a_2=20a_1$ or $a=c\varepsilon a_1$, but we do not want the leading coefficient to be of the order of ε, so we choose the first alternative. Lastly, from the coefficient of u_2 we obtain

$$a_4=\frac{3a_2a_3}{20a_1}=\frac{3}{20a_1}20a_1\cdot 10a_1=30a_1$$

So that we get $a_2=10a_1$, $a_3=20a_1$, and $a_4=30a_1$ for the existence of nontrivial (smooth function of ε) symmetries of PKdV.

It is interesting that the same values of these constants were obtained by Kodama from the condition of existence of Hamiltonian structure and Birkoffian transformation. Once the constants are fixed, one can proceed for higher order symmetries and obtain the recursion operator. We do not follow this path, but try to analyze the singularity structure of the solution manifold with the help of a Painlevé analysis.

3. PAINLEVÉ APPROACH TO PKdV

For the Painlevé analysis we set (Weiss *et al.*, 1983; M. D. Kruskal, personal communication)

$$u(Xt)=\sum \phi_{u_j}^{\alpha+j}(x, t)$$

where $\phi(x, t)=0$ defines the singularity manifold. To obtain information about α, we set $u\approx u_0\phi^\alpha$ in (1) and match the exponents of leading terms. There are several possibilities, gving rise to different branches of analysis.

(i) When u_3 and $6uu_1$ match

$$\alpha=-2; \qquad u_0=-2\phi_1^2$$

(ii) When u_3 and $\varepsilon a_1 30u^2u_1$ match

$$\alpha=-1; \qquad u_0^2=-(1/5\varepsilon a_1)\phi_1^2$$

(iii) When $6uu_1$ and $\varepsilon a_1 u_5$ match

$$\alpha=-4; \qquad u_0=-\varepsilon a_1 280\phi_1^4$$

(iv) When u_5 and $10uu_3$ or $20u_1u_2$ match

$$\alpha=-2; \qquad u_0=-3\phi_1^2$$

(v) When u_5 and $30u^2u_1$ match

$$\alpha=-2; \qquad u_0=\pm 2\sqrt{3}i\phi_1^2 \tag{11}$$

In the following we discard (ii) and (v) because (ii) gives a singular nature as $\varepsilon\to 0$ and (v) leads to complex value of u_0, while all our quantities are real. If we now substitute

$$u_0 = \sum u_j(X_1 t)\phi^{\alpha+j}(x, t)$$

in (1) and equate the coefficients of equal powers of ϕ, then we get a recursion relation for u_j (we write this out in the Appendix, due to its complex and elaborate nature). In the sequel we refer to this equation as (A1). To fine the resonance positions we now set $\alpha=-2$ and $u_0 = K\phi_1^2$, to calculate the coefficient of u_j, which yields

$$(j+1)[a_1(j^3-15j^2+86j-240)+a_2K(j-4)2a_3K]=0 \tag{12}$$

For $j=6$ we get

$$a_1k^2+6K(a_3+2a_2)+360a_1=0$$

or

$$rk^2+6K(q+2p)+360=0$$

where we have set $r=a_4/a_1$, $q=a_3/a_1$, and $p=a_2/a_1$.

For a particular value of K, if we impose the conditions that we will have resonances only at positive integral values of j, then p, q, r will be restricted. In general there really exist many possibilities. By a detailed analysis of equations (12) and (13) we see that a possible parametrization of (p, q, r) is

$$r=3p; \qquad q=30-p; \qquad p=95+1 \tag{14}$$

where S is a positive integer. For example, if we set $S=1$, then $p=10$, $q=20$, $r=30$, which is actually the set of values determined by our symmetry analysis.

Let us now proceed to the actual determination of the resonance positions. Consider first the branch $\alpha=-2$, $u_0=-2\phi_x^2$ for which coefficient of u_j in (A1) leads to

$$(j+1)(j-2)(j-5)(j-6)(j-8)=0 \tag{15}$$

so that the resonances are at $j=-1$, 2, 5, 6, 8. It is interesting to observe that this set is quite distinct from the set of resonances of the usual KdV equation $(-1, 4, 6)$. Then, from the recursion relation (A1) we can easily obtain

$$j=0;\qquad u_0=-2\phi_x^2$$

$$j=1;\qquad u_1=2\phi_2$$

$$j=2;\qquad \text{identically satisfied}$$

$$j=3;\qquad u_3=-\frac{\phi_2^3}{2\phi_1^4}+\frac{\phi_2\phi_3}{\phi_1^3}-\frac{\phi_4}{2\phi_1^2}-\frac{u_{2x}}{\phi_1} \tag{15a}$$

$$j=4;\qquad \phi_t=4\phi_3+6\phi_1u_2-\frac{3\phi_2^2}{\phi_1}+\varepsilon a_1\left(-\frac{5\phi_2^2\phi_3}{\phi_1^2}-\frac{5\phi_2\phi_4}{\phi_1}\right.$$
$$\left.+\phi_5-30\frac{\phi_2^2u_2}{\phi_1}+40\phi_3u_2+30\phi_1u_2^2+10\frac{\phi_3^2}{\phi_1}\right) \tag{15b}$$

$$j=5;\qquad -4\phi_1\phi_{1t}-2\phi_2\phi_t=[P_{2xx}-3\phi_1P_{3x}+Q_4]_x-P_3\phi_3-\phi_5Q_1 \tag{16}$$

where we have used the notation in the Appendix to write the equations in short form. Using the explicit forms obtained from the recursion relation (A1), we find

$$(P_{2xx}-3\phi_1P_{3x}+Q_4)_x$$
$$-6\phi_2\phi_3-4\phi_1\phi_4+\frac{6\phi_2^3}{\phi_1}$$
$$-24\phi_1\phi_2u_2+\varepsilon a_1\left[18\phi_2\phi_5+32\phi_3\phi_4-4\phi_1\phi_8\right.$$
$$\left.-\frac{60\phi_2^5}{\phi_1^3}+180\frac{\phi_2^3\phi_3}{\phi_1^2}-50\frac{\phi_2^2\phi_4}{\phi_1}-120\frac{\phi_2\phi_3^2}{\phi_1}\right]$$
$$+60u_2\frac{\phi_2^3}{\phi_1}-60\phi_2\phi_3u_2-40\phi_1\phi_4u_2-120\phi_1\phi_2u_2^2-P_3\phi_2-\phi_1Q_5$$
$$-2\phi_2\phi_3-12\phi_1\phi_2u_2+\varepsilon a_1\left(-12x\phi_3\phi_4\right.$$
$$\left.-20\phi_2\phi_3u_2-10\frac{\phi_3\phi_2^3}{\phi_1^2}+20\frac{\phi_2\phi_3^2}{\phi_1}-60\phi_1\phi_2xu_2^2\right) \tag{17}$$

$$j=6; \qquad 2\phi_{2t} = P_{3xxx} + Q_{5x}$$
$$P_3 = 2\varepsilon a_1\phi_4 + 20\varepsilon a_1\phi_2 u_2 + 2\phi_2 \tag{18}$$
$$Q_5 = 6u_1u_2 + 30\varepsilon a_1 u_1 u_2^2 - 10\varepsilon a_1 u_{1x} u_{2x}$$

In writing equations (16)-(18) we have already had recourse to the truncation

$$u = u_0\phi^{-2} + u_1\phi^{-1} + u_2 \tag{18a}$$

to shorten the structure. Of course, when we set $u_3 = u_4 = u_5 = \cdots = 0$, equation (15a) imposes a restriction on ϕ and u_2, the consequences of which will be discussed in the following.

At $j=7$, we get an equation involving only $u_3, u_4, \ldots$, which have been all set to zero, so it is trivial.

At $j=8$, after the trunctation, we get the equation

$$u_{2t} = u_{2xxx} + 6u_2u_{2x} + a_1\varepsilon(u_{2xxxxx} + 10u_2u_{2xxx} + 20u_2u_{2xx} + 30u_2^2u_{2x}) \tag{19}$$

So that u_2 is a solution of the PKdV.

Our main concern is now to prove the consistency of the overdetermined set obtained above after trunctation, so that (18a) can be interpreted as a Backlund transformation (BT).

Let us start with $u_3 = 0$, or, from (15a),

$$u_{2x} = \left(-\frac{\phi_2^3}{2\phi_1^3} + \frac{\phi_2\phi_3}{\phi_1^2} - \frac{\phi_4}{2\phi_1}\right) \tag{20}$$

and differentiate (15b) to get

$$\phi_{1t} = \phi_4 + 6\phi_2u_2 + \varepsilon a_1\left(-\frac{40\phi_2^3\phi_3}{\phi_1^3} - \frac{5\phi_2\phi_5}{\phi_1} - \frac{5\phi_3\phi_4}{\phi_1} + \phi_6 + 10\phi_4u_2 \right.$$
$$\left. + 30\phi_2\phi_2^2 + \frac{15\phi_2^5}{\phi_1^4} + \frac{15\phi_2^2\phi_4}{\phi_1^2} + 20\phi_2\frac{\phi_3^2}{\phi_1^2}\right) \tag{21}$$

Now construct

$$-4\phi_1\phi_{1t} - 2\phi_2\phi_t$$
$$-4\phi_1\phi_4 - 36\phi_1\phi_2u_2 + 6\frac{\phi_2^3}{\phi_1} - 8\phi_2\phi_3$$
$$+\varepsilon a_1\left[170\frac{\phi_2^3\phi_3}{\phi_1^2} + 18\phi_2\phi_5 + 20\phi_3\phi_4 - 4\phi_1\phi_6 - 40\phi_1\phi_4u_2\right.$$
$$-180\phi_1\phi_2u_2^2 - 60\frac{\phi_2^5}{\phi_1^3} - 50\frac{\phi_2^2\phi_4}{\phi_1}$$
$$-100\frac{\phi_3^2\phi_2}{\phi_1} + 60\frac{\phi_2^3u_2}{\phi_1} - 80\phi_2\phi_3u_2 \tag{22}$$

which is nothing but equation (16). So (15a), (15b), and (16) are compatible. We now consider equation (18). By one integration we obtain from (18)

$$2\phi_{1t} = P_{3xx} + Q_5$$

or

$$\phi_{1t} = \phi_4 + 6\phi_2 u_2 + \varepsilon a_1(\phi_6 + 10\phi_4 u_2 + 10\phi_3 u_{2x} + 10\phi_2 u_{2xxx} + 30\phi_2 u_2^2) \quad (23)$$

Integrating again,

$$\phi_t = 4\phi_3 + 6\phi_1 u_2 - \frac{3\phi_2^2}{\phi_1} + \varepsilon a_1 \phi_5 + 10\phi_2\left(-\frac{\phi_2^3}{2\phi_1^3} + \frac{\phi_2\phi_3}{\phi_1^2} - \frac{\phi_4}{2\phi_1}\right)$$
$$+ 30\phi_1 u_2^2 + 40 u_2 \phi_3 - 30\frac{\phi_2^2}{\phi_1} u_2$$
$$-10\int\left(\frac{3}{2}\frac{\phi_2^5}{\phi_1^4} - \frac{5\phi_2^3\phi_3}{\phi_1^3} + \frac{4\phi_2\phi_3^2}{\phi_1^2} + \frac{3\phi_2^2\phi_4}{2\phi_1^2} - \frac{2\phi_3\phi_4}{\phi_1}\right) dX \quad (24)$$

but this integrand is

$$= \frac{\partial}{\partial x}\left(-\frac{1}{2}\frac{\phi_2^4}{\phi_1^3} + \frac{3}{2}\frac{\phi_2^2\phi_3}{\phi_1^2} - \frac{\phi_3^2}{\phi_1}\right)$$

Rearranging and multiplying by ϕ_1^2, we arrive at [using equation (20)]

$$\phi_1(\phi_1\phi_t - 4\phi_1\phi_3 - 6\phi_1^2 u_2 + 3\phi_2^2)$$
$$= \varepsilon a_1(\phi_1^2\phi_5 - 5\phi_2^2\phi_3 - 5\phi_1\phi_2\phi_4 + 10\phi_1\phi_3^2$$
$$+ 30\phi_1^3 u_2^2 + 40\phi_1^2 u_2\phi_3 - 30\phi_1\phi_2^2 u_2) \quad (25)$$

which is nothing but equation (15b) in different form.

Lax pair

For the derivation of the Lax pair, we start from equations (20) and (15b). If we define the Schwarzian derivative,

$$\{\phi, x\} = \frac{\partial}{\partial x}\left(\frac{\phi_2}{\phi_1}\right) - \frac{1}{2}\left(\frac{\phi_2}{\phi_1}\right)^2 \quad (26)$$

then (15b) can be written as

$$\frac{\phi_t}{\phi_1} - \{\phi_{1,x}\} + 6\lambda = \varepsilon a_1\left[\{\phi, x\}_{xx} + 2u_2\{\phi_2 x\} + \frac{5}{2}\{\phi x\}^2 - 8\lambda\{\phi, x\} + 30\lambda^2\right] \quad (27)$$

As pointed out by Weiss, this form immediately suggests that this equation can be written as a linear equation. Indeed, if we set $\phi = V^2$ in (15b), then

$$V_t = (2u_2\partial_x + u_{2x} - 4\lambda\partial_x)V + \varepsilon a_1(32\lambda^2\partial x - 8u_2\lambda\partial_x - 10u_2^2\partial_x - 4\lambda u_{2x} - u_{2xx})V + \varepsilon a_1(-22u_2u_{2x} + 2u_{2x}\partial_x)V \tag{28}$$

Now, if in equation (20) we set $\phi_1 = V^2$, then

$$u_{2x} = -\phi_2^3/2\phi_1^3 + \phi_2\phi_3/\phi_1^2 - \phi_4/2\phi_1 \tag{29}$$

can be written as

$$u_{2x} = -(V_3/V - V_2V_1/V^2) \tag{30}$$

which can be integrated once

$$u_2 = -V_2/V + \lambda \qquad (\lambda \text{ a constant}) \tag{31}$$

or

$$(u_2 - \lambda)V = V_2$$

or

$$\partial^2 V_2/\partial x^2 = (u_2 - \lambda)V \tag{32}$$

which is nothing but the Schrödinger equation (x part of the Lax pair).

4. OTHER BRANCHES OF THE PAINLEVÉ ANALYSIS

As observed previously, for $\alpha = -2$ we also have another situation, for which $u_0 = -6\phi_1^2$. In that case $K = -6$ and the equation governing the positions of the resonances is

$$(j+1)(j-6)(j^3 - 15j^2 + 26j + 240) = 0 \tag{33}$$

yielding resonances at

$$j = -3, -1, 6, 8, 10 \tag{34}$$

Once again $j = -1$ corresponds to the arbitrariness of $\phi(x, t)$, but $j = -3$ is of no use. Thus, the number of arbitrary functions that can enter the expansion is one less than the number of resonances, in contradiction to the requirement of the Cauchy-Kawalevska theorem. Similarly, for the branch $\alpha = -4$, $u_0 = -a_1\varepsilon 280\phi_1^4$ we get a resonance at $j = -1$. Thus all other branches that can occur do not possess the Painlevé property, but for the branch $u_0 = -2\phi_1^2$, $\alpha = -2$ the equation is completely integrable, since it is possible to deduce the Lax pair explicitly.

5. PAINLEVÉ TEST FOR PERTURBED NLS EQUATION

The equation under consideration is written

$$q_t = +iq_{xx} + 2iq^2 q + (a_1 q_{xxx} + a_3 q^2 q_x^*)$$
$$q_t^* = -iq_{xx}^* - 2iq^2 q^* + \varepsilon(a_1 q_{xxx}^* + a_3 q^{*2} q_x) \tag{35}$$

If we proceed as before by finding the leading exponent to be -1, we substitute

$$q_j = \sum_{j=0}^{\infty} u_j \phi^{j-1}; \qquad q_j^* = \sum_{j=0}^{\infty} V_j \phi^{j-1} \tag{36}$$

Equating coefficients of the same power of ϕ, we are led to recursion relations,

$$\begin{aligned}
&u_j - 3t + (j-3)u_j - 2_t \\
&\quad = j(j-2)(j-3)\phi_x^2 u_{j-1} \\
&\qquad + j(j-3)\Bigg[2\phi_x u_{j-2,x} + \phi_{xx} u_{j-2} + iu_{j-3xx} \\
&\qquad + \sum_m \sum_n \{2iu_m u_n V_{j-m-n-1}\} + \varepsilon a_1[(j-1)(j-2)(j-3)\phi_1^3 u_j \\
&\qquad + (j-2)(j-3)(3u_{j-1x}\phi_x^2 + 3u_{j-1}\phi_x\phi_{xx}) + (j-3)(3u_{j-2xx}\phi x_x \\
&\qquad + 3u_{j-2x}\phi_{xx} + u_{k-2}\phi_{xxx}) + u_{j-3xxx}] \\
&\qquad + \varepsilon a_3 \left[\phi_x \sum_m \sum_n u_m u_n V_{j-m-n}(J-m-n-1) + \sum_m \sum_n u_m u_n x V_{j-m-n-1,x} \right]\Bigg]
\end{aligned} \tag{37}$$

$$\begin{aligned}
&V_{j-3,t} + (j-3)V_{j-2}\phi_t \\
&\quad = -i(j-2)(j-3)\phi_x^2 V_{j-1} - j(j-3)(2\phi_x V_{j-2x} + \phi_{xx} V_{j-2}) \\
&\qquad - iV_{j-3xx} - 2i \sum_m \sum_n V_m V_n u_{j-m-n-1} \\
&\qquad + \varepsilon a_1[(j-1)(j-2)(j-3)\phi_x^3 V_j + (j-2)(j-3)(3\phi_x^2 V_{j-1x} + 3\phi_x\phi_{xx} V_{j-1}) \\
&\qquad + (j-3)(3V_{j-2xx}\phi_x + 3V_{j-2x}\phi_{xx} + V_{j-2}\phi_{xxx}) + V_{j-3xxx}] \\
&\qquad + \varepsilon a_3 \left[\phi_x \sum_m \sum_n V_m V_n u_{j-m-n}(j-m-n-1) + \sum_m \sum_n V_m V_n u_{j-m-n-1x} \right]
\end{aligned} \tag{38}$$

Setting $j = 0$, we get

$$u_0 V_0 = -(6a_1/a_3)\phi_x^2 \tag{39}$$

Taking coefficients of u_j and V_j,

$$\varepsilon a_1[(j-1)(j-2)(j-3)\phi_x^3]u_j+\varepsilon a_3[\phi_x u_0^2 V_i(j-1)-2u_0 V_0\phi_x u_j]$$
$$=\text{terms containing } U_j,\ V_j \text{ lower order} \tag{40}$$

$$\varepsilon a_1[(j-1)(j-2)(j-3)\phi_x^3 V_j+\varepsilon a_3[\phi_x V_0^2 u_j(j-1)-2u_0 V_0\phi_x V_j]$$
$$=\text{terms containing } u_j,\ V_j \text{ lower order} \tag{41}$$

so that we have the system matrix (Goldstein and Infeld, 1984)

$$T=\begin{pmatrix} a_1(j-1)(j-2)(j-3)\phi_x^3-2a_3u_0V_0\phi_x, & a_3\phi_x u_0^2(j-1)\\ a_3\phi_x V_0^2(j-1), & a_1(j-1)(j-2)(j-3)\phi_x^3-2a_3u_0V_0\phi_x\end{pmatrix}$$

If we use (39) for $u_0 V_0$, then $\det T=0$ leads to

$$j(j+1)(j-3)(j-4)(j^2-6j+17)=0 \tag{42}$$

so that we have resonances at $j=0, -1, 3, 4$, but those from the last factor are not integral. For $j=1$ in the recursion relation,

$$2i\phi_x^2 u_0\left(1-\frac{6a_1}{a_3}\right)+6\varepsilon a_1\phi_x(u_{0x}\phi_x u_0\phi_{xx})$$
$$+\varepsilon a_3(u_0^2 V_{0x}-2\phi_x u_0 V_0 u_1)=0 \tag{43}$$

$$-2i\phi_x^2 V_0\left(1-\frac{6a_1}{a_3}\right)+6\varepsilon a_1\phi_x(V_{0x}\phi_x+V_0\phi_{xx})$$
$$+\varepsilon a_3(V_0^2 u_{0x}-2\phi_x u_0 V_0 V_1)=0 \tag{44}$$

Multiplying (43) by u_0 and (44) by V_0, if we add and subtract, then we get

$$4i\phi_x^2 u_0 V_0\left(1-\frac{6a_1}{a_3}\right)+6\varepsilon a_1\phi_x^2(u_{0x}V_0-u_0V_{0x})+\varepsilon a_3 V_0 u_0 x$$
$$[-(u_{0x}V_0-V_{0x}u_0)+2\phi_x(u_0V_1-V_0u_1]=0 \tag{45}$$

Since ε is small, we demand $1-6a_1/a_3=0$, so that equation (   ) is written as

$$u_0V_1+V_0u_1=\phi_{xx} \tag{46}$$

and we also get

$$V_1u_0-V_0u_1=(u_{0x}V_0-V_{0x}u_0)/\phi_x \tag{47}$$

along with

$$u_0 V_0 = -\phi_x^2 \tag{48}$$

Putting $j=2$ in the recursion relations, we obtain

$$\begin{aligned}
-u_0\phi_t &= -i(2u_{0x}\phi_x + u_0\phi_{xx}) + 2iu_0^2 V_1 + 4iu_0 V_0 u_1 \\
&\quad + \varepsilon a_1[-(3u_{0xx}\phi_x + 3u_{0x}\phi_{xx} + u_0\phi_{xxx})] \\
&\quad + \varepsilon a_3(\phi_x u_0^2 V_2 - \phi_x u_1^2 V_0 + u_0^2 V_{1x} + 2u_0 u_1 V_{0x}) - V_0\phi_t \\
&= i(2V_{0x}\phi_x + V_0\phi_{xx}) - 2iV_0^2 u_1 - 4iu_0 V_0 V_1 \\
&\quad + \varepsilon a_1[-(2V_{0xx}\phi_x + 3V_{0x}\phi_{xx} + V_0\phi_{xxx})] \\
&\quad + \varepsilon a_3(\phi_x V_0^2 u_2 - \phi_x V_1^2 u_0 + V_0^2 u_{1x} + 2V_0 V_1 u_{0x})
\end{aligned} \tag{49}$$

Now, since we want to truncate the expansions at constant level, we set u_j, $V_j=0$, $j\geq 2$. This yields, from (47) and (49),

$$u_{0x} V_0 - u_0 V_{0x} = \lambda\phi_x^2 \tag{50}$$

$$2\phi_t/\phi_x = \varepsilon a_1(2\psi_1 - \psi^2 + \tfrac{3}{2}\lambda^2) \tag{51}$$

where

$$\psi = \phi_{xx}/\phi_x; \qquad u_1 V_1 = \tfrac{1}{4}(\lambda^2 - \psi^2) \tag{52}$$

Now for $j=3$ (after the truncation is taken into account) we get

$$\begin{aligned}
u_{0t} &= iu_{0xx} + 2iu_1^2 V_0 + 4iu_0 u_1 V_1 + \varepsilon a_1 u_{0xxx} \\
V_{0t} &= -iV_{0xx} - 2iV_1^2 u_0 - 4iV_0 V_1 u_1 + \varepsilon a_1 V_{0xxx}
\end{aligned} \tag{53}$$

from which it is easy to deduce

$$2\phi_{it}/\phi(x) = \varepsilon a_1(2\psi_2 - \psi^3 + \tfrac{3}{2}\lambda^2\psi) \tag{54}$$

It is now interesting to observe that equation (54) is nothing but the derivative of (   ); that is,

$$\frac{\partial}{\partial x}\left[\frac{2\phi_t}{\phi_x} - \varepsilon a_1\left(2\psi_1 - \psi^2 + \frac{3}{2}\lambda^2\right)\right] = 0$$

so that they are compatible. Now, by subtracting upon multiplication by V_0 and u_0, we can deduce from (53) and (50)

$$\frac{V_0^2}{\phi_x^2}\frac{\partial}{\partial t}\left(\frac{u_0}{V_0}\right) = i\left(\psi^2 - 2\psi_1 - \frac{3}{2}\lambda^2\right) + \varepsilon a_1\lambda\left(2\psi^2 - 4\psi_1 + \frac{\lambda^2}{4}\right) \tag{55}$$

A similar consideration at $j=4$ leads to

$$(u_1 V_1)_t = i\frac{\partial}{\partial x}(V_1 u_{1x} - u_1 V_{1x}) + \varepsilon a_1[(u_1 V_1)_{xxx} - 3(u_{1x}V_{1x})_x]$$
$$+6\varepsilon a_1 (u_1 V_1)_x u_1 V_1 \tag{56}$$
$$V_1^2 \frac{\partial}{\partial t}\left(\frac{u_1}{V_1}\right) = i[(u_1 V_1)_{xx} - 2u_{1x}V_{1x}] + 4i(u_1 V_1)^2$$
$$+\varepsilon a_1(V_1 u_{1xxx} - u_1 V_{1xxx}) - 6\varepsilon a_1 u_1 V_1(V_1 u_{1x} - u_1 V_{1x})$$

Putting the values of $u_1 V_1$, etc., we get

$$\psi_t = i\lambda\left[\frac{\{\phi_x\}_x}{\psi} + \frac{\psi_1}{2}\right] + \varepsilon a_1\left[\{\sigma_x\}_{xx} + \psi\{\phi_x\}_x - \frac{3\lambda^2}{\psi}\right.$$
$$\left.\times\{\phi x\}_x - \frac{1}{2}\lambda^2 \quad {}_1 - \frac{1}{2} \quad {}^2 \quad {}_1 + \psi_1^2 + \frac{3}{4}\psi_1^2\lambda\right] \tag{57}$$

For compatibility with (52) we should have

$$\lambda\left[\frac{\{\phi_x\}_x}{\psi} + \frac{\psi^2}{4} + \frac{1}{2}\{\phi_x\}\right] = 0, \qquad \lambda^2\{\phi x\}_x = 0 \tag{58}$$

$\{\phi, x\}$ being the Schwarzian derivative

$$\{\phi, x\} = \frac{\partial}{\partial x}\left(\frac{\phi_2}{\phi_1}\right) - \frac{1}{2}\left(\frac{\phi_2}{\phi_1}\right)^2$$

In the above computation λ is a constant of integration, which also can be a function of time. It is easily observed that even then equations (50)-(52) and (54) remain unchanged, except that we now write $\lambda(t)$ instead of λ. We have an extra equation

$$\psi\psi_t - \lambda\lambda_t = -i\lambda(\psi_2 - \psi\psi_1) + \varepsilon a_1(\psi\psi_3 - \tfrac{3}{2}\lambda^2\psi_2$$
$$+\tfrac{3}{4}\lambda^2\psi\psi_1 + \tfrac{3}{2}\lambda^2\psi\psi_1 - \tfrac{3}{2}\psi^3\psi_1) \tag{59}$$

From equation (   ) it again follows that $\lambda^2\{\phi x\}_x = 0$, so either $\lambda = 0$ or $\{\phi x\}_x = 0$.

If we set $\lambda = 0$, then we deduce

$$i(\psi^2 - 2\psi_1) = -(i/\psi)(2\psi_2 - \psi^3)$$

immediately leading to $\psi_2 - \psi\psi_1 = 0$, which is nothing but $\{\phi x\}_x = 0$, again fixing ϕ. Thus we conclude that the perturbed NLS equation does not pass the Painlevé test, in contrast to the perturbed KdV equation. We now show that though our equation (PNLS equation) does not conform to the Painlevé criterion, yet it is possible to deduce its conservation laws and the corresponding Lax pair via the approach of Chen and Liu.

The Converservation Laws and Lax Pair

Let us now linearize equation (35) by setting $q \to q + \varepsilon\phi$, $q^* \to q^* + \varepsilon\psi$; we then get

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}_t = M \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

with M given as

$$M = \begin{pmatrix} -i\partial^2 - 4i|q|^2 + \varepsilon(a_1\partial^3 - a_3 2qq_x^*), & 2iq^* + \varepsilon a_3 q^* \partial^2 \\ -2iq^2 + \varepsilon a_3 q^2 \partial, & i\partial^2 + 4i|q|^2 + \varepsilon(a_i\partial^3 - 2a_3 q^* q) \end{pmatrix} \tag{60}$$

We now substitute (Iino *et al.*, 1982)

$$\begin{aligned} \phi &= \exp\left(kx + wt + \int_{-\infty}^{x} T\,dx\right) \\ \psi &= A \exp\left(kx + wt + \int_{-\infty}^{x} T\,dx\right) \end{aligned} \tag{61}$$

and ensure that A, T are analytic functions of k to be expanded as

$$T = \sum_{n=0}^{\infty} k^{-n} T_n; \qquad A = \sum_{n=0}^{\infty} k^{-n} A_n \tag{62}$$

Equations obtained for A_n, T_n after we substitute (61) in equation (60) can be solved recursively for the coefficients T_n and A_n. Let us assume that each T_n, A_n have perturbed and nonperturbed contributions written as $T_n = T_n^0 + \varepsilon T_n^1$ and $A_n = A_n^0 + \varepsilon A_n^1$. We then get the results shown in Table I.

Now, as per the ansatz of Chen and Liu, we consider M to be the time part of the Lax pair. For the space part we observe that these $T_n^{0,1}$ can lead to solution sets of (   ) by $\delta T_n/\delta q$, $\delta T_n/\delta q^*$, which can be recursively generated through an operator L of the form

$$L = \begin{pmatrix} \partial + a & b \\ c & -\partial + d \end{pmatrix} = L_0 + L_1 \tag{63}$$

$$L_0 = \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix}, \qquad L_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We now impose the condition that $L_t = [A, L]$ holds up to first order in ε,

Table I

	T_n^0	A_n^0
$n=0$	0	0
$n=1$	$-2qq^*$	0
$n=2$	$2q^*q_x$	q^2
$n=3$	$-2q^*q_{xx}-2(qq^*)^2$	—
	T_n^1	A_n^1
$n=0$	$3ia_1qq^*$	0
$n=1$	$3a_1(q_x^*q-q^*q_x)+ia_3qq_x^*$	$\frac{1}{2}ia_3q^2$
$n=2$	$-3a_1[q_xq_x^*-q^*q_x^*-(qq^*)^2]$ $+3ia_1(qq^*)^2$ $+ia_1(q_{xx}q^*-q_xq_x^*+qq_{xx}^*)$	$(3a_1-a_3)qq_x$

which then leads to the following equations for the coefficients (a, b, c, d):

$$\begin{aligned}
a_t\psi &= (ia\partial^2 - i\partial^2 a)\psi + \varepsilon(a_1\partial^3 a - a_1 a\partial^3)\psi t(M\partial - \partial M)\psi \\
c_t\psi &= i(\partial^2 c + c\partial^2)\psi + \varepsilon a_1(\partial^3 c - c\partial^3)\psi + (N^*\partial^* + \partial N^*)\psi \\
b_t\psi^* &= -i(\partial^2 b + b\partial^2)\psi^* + \varepsilon a_1(\partial^3 b - b\partial^3)\psi^* - (N\partial + \partial N)\psi^* \\
d_t\psi^* &= i(\partial^2 d - d\partial^2)\psi^* + \varepsilon a_1(\partial^3 d - d\partial^3)\psi^* + (\partial M^* - M^*\partial)\psi^*
\end{aligned} \tag{64}$$

For the actual solution we again assume that $a=a^0+\varepsilon a^1$, $b=b^0+\varepsilon b^1$, $c=c^0+\varepsilon c^1$, etc., and regard q, q^* as small quantities; so we use, as in the mode coupling approximation,

$$q=\sum e^{lx+il^2t}q_l, \qquad \psi = e^{kx-ik^2t}\psi_k \tag{65}$$

Then if we Fourier decompose the coefficients a^0, a^1, etc, as

$$a^0=\sum_{lm} a_{lm}^0 \exp[(l+m)x+i(l^2-m^2)t] \tag{66}$$

we obtain

$$a_{lm}^0 = 2q_1q_m^*\left(1+\frac{m+k}{1+k}\right)$$

$$a_{lm}^1 = -ia_1q_1q_m^*\left(1+\frac{m+k}{1+k}\right)\left(1+2k+2m+\frac{m^2+mk+k^2}{1+k}\right)+2a_3q_1q_m^*$$

$$b_m^0 = -q_1^*q_m^*\frac{1+m+2k}{(1+k)(k+m)}$$

$$b^1_{lm} = -\frac{1}{2} ia_1 b^0_{lm} \frac{1+m}{(1+k)(m+k)} (l^2+m^2+2lm+3lk+3mk+3k^2) + \frac{la_3}{2} k\left(\frac{1}{l+k}+\frac{1}{m+k}\right) q^*_l q^*_m \tag{67}$$

$$c^0_{lm} = q_l m_m \left(\frac{1}{l+k}+\frac{1}{m+k}\right)$$

$$c^1_{lm} = \frac{a_1 i}{2} c^0_{ml} \frac{(l+m)}{(l+k)(m+k)} (l^2+m^2+2lm+3lk+3mk+3k^2) + \frac{ia_3}{2} q_1 q_m k\left(\frac{1}{l+k}+\frac{1}{m+k}\right)$$

Similar results can also be deduced for d^0, d^1. It is interesting to note that though we deduced the values by assuming the Fourier decomposition, it is independent of it, since the last expressions can be interpreted as operators on the fields q, ψ, etc. As an example, we easily observe that

$$a_1(1+k)+(m+k)+\frac{(m+k)^2-mk}{1+k}\psi_k q_1 q^*_m \tag{68}$$

is nothing but

$$a_1[q^*\,\partial x(q\psi)+q\,\partial x(q^*\psi)+q^*_{xx}D^{-1}(\psi q)+q^*_x D^{-1}(\psi_x q)+q^* D^{-1}(q\psi_{xx})] \tag{69}$$

So if we assume that this holds for all, then dropping ψ, we can write the operator expression

$$a_1[q^*\,\partial x(q)+q\,\partial x(q^*)q^*_{xx}D^{-1}(q)+q^*_x D^{-1}(q\,\partial x)+q^* D^{-1}(q\,\partial_{xx})] \tag{70}$$

Similar considerations hold for the other coefficients. Thus although the peturbed NLS equation does not pass the Painlevé analysis, it is possible to find a Lax pair via the Chen–Liu approach.

6. DISCUSSION

In the above analysis we have considered the questions of complete integrability and existence of an infinite number of symmetry conservation laws of perturbed KdV and perturbed NLS systems. The arbitrary constants occurring in the perturbing term could be fixed by demanding the smooth behavior of Lie–Backlund symmetry with respect to the perturbation parameter. It is interesting to note that the same set of values arises when we apply Painlevé analysis to PKdV. Though various branches exist, only one branch has the full Painlevé behavior, where we prove the consistency of the overdetermined set of equations. We then show that it is possible to deduce the Lax pair from these Painlevé equations. Lastly, it is important to note that the set of resonances in our PKdV is different from those of the ordinary KdV equation, yet our equation ( ) goes smoothly to those of Weiss et al. as $\varepsilon \to 0$. The same approach, when applied to the case of the perturbed NLS equation, yields that this perturbed system does not conform to the Painlevé criterion at all. But it is really surprising to observe that one can find a Lax pair for the perturbed NLS system through the use of many conservation laws and their recurrence relation following the approach of Chen and Liu. Perhaps this another example that Painlevé analysis is not a necessary and sufficient condition for the test of complete integrability of nonlinear systems. In the Appendix we give another example of a Hamiltonian KdV system that is still believed to be completely integrable, has infinite symmetry recursion operator and Hamiltonian structure, but does not pass the Painlevé test.

APPENDIX

We discuss a new nonlinear pde known as the "Hamiltonian KdV" system discussed by Olver (1987), which was deduced from the basic equations of hydrodynamics by a modified perturbation theory respecting the symplectic structure at every order. The equation under consideration reads (Olver, 1987)

$$u_t + u_x + \frac{3}{2}\alpha u u_x + \frac{\beta}{6} u_{xxx} + \frac{\alpha\beta}{16}(u^2)_{xxx} + \frac{15}{32}\alpha^2 u^2 u_x = 0 \tag{A1}$$

Proceeding as before, we find the leading exponent to be -2. So we set $u = \sum_{j=0}^{\infty} u_j \phi^{j-2}$ in (A1), leading to the recursion relation,

$$\begin{aligned}
&(j-6)u_{j-4}\phi_t + u_{j-5,t} + (j-6)u_{j-4}\phi_x + u_{j-3x} \\
&\quad + \tfrac{3}{2}\alpha[u_m u_{j-m-2}(j-m-4)\phi_x + u_m u_{j-m-3x}] \\
&\quad + \tfrac{1}{6}\beta[(j-4)(j-5)(j-6)u_{j-2}\phi_x^3 + (j-5)(j-6)(3u_{j-3x}\phi_x^2 + 3u_{j-3}\phi_x\phi_{xx}) \\
&\quad + (j-6)(3\phi_x u_{j-4xx} + 3\phi_{xx}u_{j-4x} + u_{j-4}\phi_{xxx}) + u_{j-5xxx}] \\
&\quad + \tfrac{1}{16}\alpha\beta\{(j-4)(j-5)(j-6)u_m u_{j-m}\phi_x^3 + (j-5)(j-6)[3\phi_x^2(u_m u_{j-m-1})_x \\
&\quad + 3\phi_x\phi_{xx}u_m u_{j-m-1}] + (j-6)[3\phi_x(u_m u_{j-m-2})_{xx} + 3\phi_{xx}(u_m u_{j-m-2})_x \\
&\quad + \phi_{xxx}(u_m u_{j-m-2})] + (u_m u_{j-m-3})_{xxx}\} \\
&\quad + (15/32)\alpha^2[u_m u_n u_{j-m-n}(j-m-n-2)\phi_x + u_m u_n u_{j-m-n-1x}]
\end{aligned} \tag{A2}$$

From this it is easy to obtain

$$u_0 = -\frac{8\beta}{\alpha}\phi_x^2; \qquad u_1 = \frac{8\beta}{\alpha}\phi_{xx} \tag{A3}$$

The coefficient of u_j in (A2) yields resonances at $j = -1, 6, 10$. Furthermore,

$$u_3 = \frac{k}{\phi_x}\left(\frac{\psi_2}{3} - \frac{17}{21}\psi^3 - \frac{11}{7}\psi\psi_1\right) \tag{A4}$$

$$u_2 = -k\left(\frac{\psi_1}{3} + \frac{\psi^2}{12} + \frac{5}{3\alpha k}\right) \tag{A5}$$

where $\psi = \phi_2/\phi_1$. But if, following Weiss, we want to truncate the expansion at the constant level by setting $u_i = 0$, $i \geq 3$, then it is seen that one need not proceed up to $r = 6$ or 10, and we get a negative result.

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